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Short communication

Comments on a recent paper dealing with the finite-analytic method

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Introduction

The finite-analytic method was developed by Chen and co-workers in the early 1980s (Chen *et al.*, 1980) as a technique based on the decomposition of the computational domain into subdomains or computational cells. The coefficients of the governing equations were frozen in each subdomain, so that linear partial differential equations were to be solved in each computational cell subject to appropriate boundary conditions at the cell's boundaries. Chen *et al.* (1980) and Chen and Chen (1984) approximated the boundary conditions by polynomials and linear combinations of polynomials and exponentials of the co-ordinate along each boundary respectively, and employed the method of separation of variables to obtain the solution in each subdomain. Furthermore, Chen *et al.* (1980) and Chen and Chen (1984) related the coefficients in the approximate boundary conditions to the values of the vorticity at three successive nodes of any boundary in Cartesian co-ordinates, and truncated the infinite series solution given by the method of separation of variables to a finite one so that nine-point stencils resulted. Their method is, therefore, an approximate one owing to both the use of assumed boundary conditions and the use of frozen coefficients in the governing equations in each subdomain.

Suh and Benim (1989) used the finite-analytic method of Chen *et al.* (1980) with a primitive variable formulation and a pseudo-compressibility technique for the pressure, and examined two-dimensional cavity flows in Cartesian co-ordinates. Note that Chen *et al.* (1980) and Chen and Chen (1984) considered cavity flows but employed a stream function-vorticity formulation.

Sun and Militzer (1992) developed a piecewise-parabolic, finite-element method for two-dimensional advection-diffusion equations based on similar approximations to those of Chen *et al.* (1980) and Chen and Chen (1984), i.e. they employed separation of variables, except that they chose the boundary conditions in each subdomain in such a manner so as to satisfy the maximum and minimum principles of the two-dimensional advection-diffusion equation.

Civan (1995) also employed a finite-analytic technique to solve one- and two-dimensional, linear problems by employing series solutions rather than by

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linearizing the governing equations as done by, for example, Chen *et al.* (1980) and Chen and Chen (1984).

Montgomery and Fleeter (1996) employed the finite-analytic method of Chen *et al.* (1980) and Chen and Chen (1984) to analyse steady, two-dimensional, inviscid, compressible, subsonic flow in a nozzle, and employed a linearization method in each subdomain, approximate boundary conditions at the subdomain's boundaries, and the method of separation of variables to obtain continuous, albeit approximate, solutions in the whole computational domain.

The objective of this short communication is three-fold. First, it is shown that the cell boundary conditions employed by Montgomery and Fleeter (1996) are not exact. Second, by means of a simple one-dimensional example, it is illustrated that the finite-analytic method is really a piecewise-parabolic approximation which involves only three consecutive grid points and, therefore, it provides continuous but not differentiable solutions. Finally, a finite-analytic method which employs the exact boundary conditions at the cell boundaries and provides continuous and differentiable solutions is proposed.

Boundary conditions at the cell boundaries

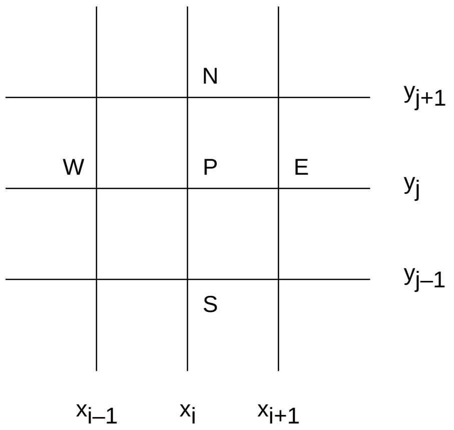
The boundary function for the $y_j + 1$ boundary (Figure 1) used by Montgomery and Fleeter (1996) was defined in their equation (13) as

$$\bar{\Phi}_{j+1}(x) = \Phi(x, y_{j+1}) = a_{j+1}[\exp(2Ax) - 1] + xb_{j+1} + c_{j+1}, \tag{1}$$

where $\overline{\Phi}$ satisfies the following partial differential equation

$$\bar{\Phi}_{xx} + C\bar{\Phi}_{yy} - 2A\bar{\Phi}_x - 2BC\bar{\Phi}_y = 0, \tag{2}$$

which may be easily obtained from equations (10) and (11) of Montgomery and Fleeter (1996).



Source: Montgomery and Fleeter (1996)

Figure 1. Computational cell

If equation (2) is assumed to apply at the $y_j + 1$ boundary, then substitution of equation (1) into equation (2) yields

$$C\bar{\Phi}_{yy}(x, y_{j+1}) - 2BC\bar{\Phi}_y(x, y_{j+1}) - 2Ab_{j+1} = 0, \tag{3}$$

which cannot, in general, be satisfied owing to the dependence of $\overline{\Phi}$ on x along the $y_j + 1$ boundary. Therefore, the statement "the boundary conditions for the computational cell shown in Figure 2 are constructed for the particular solution to (10)" (Montgomery and Fleeter, 1996, p. 63) is incorrect. Furthermore, as shown above, their equation (13) is actually the solution to

$$\bar{\Phi}_{xx} - 2A\bar{\Phi}_x + 2Ab_{j+1} = 0, \tag{4}$$

which may be obtained from equation (2) by approximating $C\overline{\Phi}_{yy}$–$2BC\overline{\Phi}_y$ by $2Ab_{j+1}$ at the y_{j+1} boundary. Furthermore, it must be noted that Chen *et al.* (1980) employed second-degree polynomials, whereas Chen and Chen (1984) employed a linear combination of a first-degree polynomial and an exponential function to approximate the cell boundary conditions, and neither of these boundary conditions satisfies the linearized vorticity equation in each computational cell.

Continuity of the solution

The approximate solution obtained with the finite-analytic method developed by Chen *et al.* (1980) is not analytical; it is only continuous. This can be easily illustrated by means of the following non-linear, second-order, ordinary differential equation

$$\frac{d^2u}{dx^2} = S(u), \tag{5}$$

subject to appropriate boundary conditions at $x = 0$ and $x = L$, so that the solution is unique. Equation (5) can be integrated analytically and reduced to a quadrature. According to the finite-analytic method of Chen *et al.* (1980), the interval $[0, L]$ is divided into subintervals and the ith computational cell is defined as $I = [x_{i-1}, x_{i+1}]$. For the sake of convenience, hereon, we will assume that an equally spaced mesh is used. Then, the finite-analytic method applied to equation (5) yields

$$u^{(I)}(x) = 2h^2S(u_i)\xi^2 + (u_{i+1} - u_{i-1} - 2h^2S(u_i))\xi + u_{i-1}, \tag{6}$$

where $\xi = (x - x_{i-1})/(x_{i+1} - x_{i-1})$, $h = x_{i+1} - x_i$ and the superscript (I) denotes the ith computational cell. Equation (6) represents a parabolic approximation to u in I and can also be obtained by quadratic interpolation.

A similar expression to equation (6) can be deduced for the $(i + 1)$th cell, and continuity of $u^{(I)}$ at the right boundary of the ith cell yields

$$u_{i+1} - 2u_i + u_{i-1} = h^2 S(u_i), \tag{7}$$

which is the standard second-order accurate finite difference discretization of equation (5). Furthermore, differentiation of equation (6) yields

$$\frac{du^{(I)}}{dx}(x_{i+1}) = S(u_i)h + \frac{u_{i+1} - u_{i-1}}{2h}, \tag{8}$$

$$\frac{du^{(I+1)}}{dx}(x_{i+1}) = \frac{u_{i+2} - u_i}{2h}, \tag{9}$$

which clearly implies that, in general, $du^{(I)}/dx\,(x_{i+1}) \neq du^{(I+1)}/dx\,(x_{i+1})$. Therefore, the finite-analytic method of Chen *et al.* (1980) only provides continuous but not differentiable solutions, and this is owing to the fact that, for one-dimensional problems, each computational cell involves three grid points.

When non-equally spaced grids are employed, $S(u_i)$ must be replaced by an appropriate weighted value involving u_{i+1}, u_i and u_{i-1}.

A differentiable finite-analytic method

As stated previously, Montgomery and Fleeter (1996) employed some approximations to the boundary conditions at the cell boundaries which do not satisfy equation (2) and their method provides a continuous, albeit approximate, solution. In order to obtain a differentiable finite-analytic method, the boundary conditions at the cell boundaries must be left unspecified and must be determined from the continuity of the solution and of its derivative normal to the cell's boundaries. This implies that equation (13) of Montgomery and Fleeter (1996) must be replaced by $\overline{\Phi}(x,y_{j+1}) = \overline{\Phi}_{j+1}(x) = F_{j+1}(x)$ where F_{j+1} is to be determined as part of the solution. Their equations (24) and (28) are valid, but their equation (25a) is to be replaced by

$$b_{jn} = \frac{1}{h\sinh(2w_n h)} \int_{-h}^{h} F_{j+1}(x)\sin[\alpha_n(x+h)]dx \tag{10}$$

where the symbols are defined in Montgomery and Fleeter (1996) and x denotes local cell co-ordinates whose origin is at the cell centre (Figure 1).

Since Montgomery and Fleeter (1996) used the method of separation of variables, the solution to their equation (12) in the cell centred at P can be written as (cf. their equation (15))

$$\Phi^{(P)} = \sum_{i=1}^{4} \Phi_i^{(P)}, \tag{11}$$

where, for example,

$$\Phi_1^{(P)} = \sum_{n=1}^{\infty} b_{1n}^{(P)} \sin[\alpha_n(x - x_i + h)] \sinh[w_n^{(P)}(y - y_j + h)], \qquad (12)$$

(x_i, y_j) denote the co-ordinates of the cell centred at point P and

$$w_n^{(P)} = \sqrt{\frac{\lambda_n + A^{(P)2} + B^{(P)2}C^{(P)}}{C^{(P)}}}, \qquad \lambda_n = \frac{n\pi}{2h}. \qquad (13)$$

Equations (11) and (12) indicate that $\Phi^{(P)}$ depends on $b_{in}^{(P)}$, i = 1,2,3,4. Furthermore, equation (11) of Montgomery and Fleeter (1996) can be written as

$$\Psi^{(P)} = \Phi^{(P)} \exp(A^{(P)}x + B^{(P)}y) - \frac{f^{(P)}(A^{(P)}x + B^{(P)}y)}{2(A^{(P)2} + B^{(P)2}C^{(P)})}, \qquad (14)$$

which represents the stream function in the Pth cell, and A, B, C and f are the locally constant coefficients which appear in equation (10) of Montgomery and Fleeter (1996).

Equation (14) clearly implies that the stream function in the Pth cell depends on $b_{in}^{(P)}$, i = 1, 2, 3, 4. In order to determine these coefficients, we impose the continuity of Ψ and $\partial\Psi/\partial n$ at the cell boundaries where n denotes the outside normal to the boundary. Thus, for example, at the y_{j+1} boundary

$$\Psi^{(P)}(x, y_{j+1}) = \Psi^{(N)}(x, y_{j+1}), \qquad (15)$$

$$\frac{\partial\Psi^{(P)}}{\partial y}(x, y_{j+1}) = \frac{\partial\Psi^{(N)}}{\partial y}(x, y_{j+1}), \qquad (16)$$

and N denotes the cell adjacent to the north of the P cell.

It is clear from equations (11), (12) and (14) that equations (15) and (16) depend on x, $b_{in}^{(P)}$ and $b_{in}^{(N)}$. These equations may be multiplied by $\sin[\alpha_n(x - x_i + h)]$ and the results integrated from $x = -h$ to $x = h$ to obtain two linear algebraic equation of the form (this is actually the condition of orthogonality)

$$\sum_{i=1}^{4} \left(\sum_{n=1}^{\infty} c_{inm}^{(P)} b_{in}^{(P)} + \sum_{n=1}^{\infty} d_{inm}^{(N)} b_{in}^{(N)} \right) = e_m^{(PN)}, \qquad (17)$$

where, for example, for equation (15)

$$c_{1nm}^{(P)} = \sinh[2w_n^{(P)}h] \exp(B^{(P)}y_{i+1}) \int_{-h}^{h} \sin[\alpha_n(x - x_i + h)] \sin[\alpha_m(x - x_i + h)] \exp(A^{(P)}x) dx, \qquad (18)$$

$$e_m^{(PN)} = \int_{-h}^{h} \left[\frac{f^{(P)}(A^{(P)}x + B^{(P)}y_{j+1})}{2(A^{(P)2} + B^{(P)2}C^{(P)})} - \frac{f^{(N)}(A^{(N)}x + B^{(N)}y_{j+1})}{2(A^{(N)2} + B^{(N)2}C^{(N)})} \right] \sin[\alpha_m(x - x_i + h)]dx. \quad (19)$$

Note that Montgomery and Fleeter (1996) used local co-ordinates with origin at the cell centre. Equations (15) and (16) together with those for the P–S, P–E and P–W interfaces where S, E and W denote the south, east and west respectively, cells surrounding the P cell, provide eight equations for $b_{in}^{(k)}$, $k = P, E, W, S$ and N, i.e. for 20 unknowns if the P, E, W, S and N cells do not have edges where boundary conditions are specified. Note that, if a P cell has edges which coincide with the domain boundaries, the number of unknowns in that cell is four minus the number of edges where the boundary conditions are specified.

Equation (17) contains series; therefore, it is, in general, impossible to obtain its analytical solution. However, if the series are truncated and a finite number of terms are retained, equation (17) may be solved to obtain $b_{in}^{(k)}$ with $n \leq N$ where N is the number of terms retained. This truncation results in an approximate expression for both $\Phi^{(P)}$ and $\Psi^{(P)}$. Furthermore, the values of $\Psi^{(P)}$ at the cell boundaries may be easily determined once $\Phi^{(P)}$ is known through equation (14). Therefore, the boundary conditions at the cell boundaries do not have to be specified or approximated as in Montgomery and Fleeter (1996).

A comparison between the differentiable piecewise finite-analytic method proposed here and the finite-analytic technique of Chen *et al.* (1990) and Chen and Chen (1984) indicates that the former is algebraically more tedious than the latter; however, the former does not require any approximation to the cell boundary conditions because these are determined from the solution. Furthermore, both methods yield series (cf. Montgomery and Fleeter, 1996, p. 66).

An advantage of the differentiable finite-analytic method presented here is that the solution is differentiable whereas that of the finite analytic method of Chen *et al.* (1984) is continuous but not differentiable. The price paid for differentiability can be easily understood from the tediousness of the method presented here. Note that the method presented here provides analytical solutions compared to that of Montgomery and Fleeter (1996) which yields difference equations because of the cell boundary conditions employed in their paper (cf. their equation (37)). Note also that the method proposed here does not have to use the cells considered above; it may also consider only one of the four cells within each macrocell (Figure 1).

It should be pointed out that standard difference equations for equation (2) usually only involve five grid points, where the finite-analytic technique of Chen *et al.* (1984) uses nine points and makes it a good candidate for problems with flows not aligned with the grid, i.e. for crosswind diffusion.

Conclusions

It has been shown that the finite-analytic method provides approximate continuous but not differentiable solutions and that, for second-order, ordinary differential equations, the method is identical to a quadratic interpolation formula. It has also been shown that the finite-analytic method can be generalized to obtain differentiable solutions, although this requirement results in a more tedious and expensive technique. The differentiability of the solution can be obtained by leaving the boundary conditions at the cell boundaries unspecified and matching the solutions and their derivatives normal to the boundaries of adjacent cells.

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